JOURNAL OF APPROXIMATION THEORY 42, 193–199 (1984)

Nearby Linear Chebyshev Approximation under Constraints*

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Communicated by E. W. Cheney

Received February 21, 1983, revised January 19, 1984

It is shown that if f is near g, the linear family L is near the linear family L', the domain X is near the domain Y, and the constraint set C is near the constraint set C', a best Chebyshev approximation to f from L on X under the constraints C is near a best Chebyshev approximation to g from L' on Y, under the constraints C'. The same problem, without constraints, was studied in [2].

Any constraint on a linear approximating function L with coefficient vector A can be formulated as $A \in C$, C a subset of the coefficient space. We assume henceforth that such a formulation has been made. C may depend on the domain, basis, or function being approximated. Examples are given later.

Let W be a compact space with metric ρ. For Y a compact subset of W and $g \in C(W)$, define

$$\| g\|_Y = \sup \{| g(x)| : x \in Y\}.$$

Let $\{\phi_1,..., \phi_n\}$ be a linearly independent subset of $C(Y)$. Let C be a subset of the set of all possible coefficient vectors for linear approximation (defined next). The coefficient vector $A = (a_1,..., a_n)$ is said to the best to $f \in C(W)$ on Y by linear combinations of $\{\phi_1,..., \phi_n\}$ under constraint C if it minimizes $\|f - \sum_{i=1}^n a_i\phi_i\|_Y$ under the constraints $(a_1,..., a_n) \in C$.

Examination of existence proofs for the unconstrained case shows that a sufficient condition for existence of a best approximation is that C be nonempty and closed. It should be noted that C is often dependent on the function f being approximated, so a global existence result may involve showing that C is nonempty and closed for every $f \in C(W)$. We need a criterion for subsets being near.

* Written on sabbatical at the University of British Columbia, Department of Mathematics, Vancouver, B.C., Canada.

0021-9045/84 $3.00

DEFINITION. Let X, Y be nonempty compact subsets of W. Define

$$\text{dist}(X, Y) = \sup\{\inf\{\rho(x, y): y \in Y\}: x \in X\},$$
$$d(X, Y) = \max\{\text{dist}(X, Y), \text{dist}(Y, X)\}.$$

For a superscript s, define

$$L^s(A) = \sum_{i=1}^{n} a_i \phi_i^s.$$

Define the parameter norm

$$\|A\| = \max\{|a_1|: 1 \leqslant i \leqslant n\}.$$

THEOREM. *Let* $\{\phi_1, ..., \phi_n\}$ *be linearly independent on* X *and* $\|\phi_i^k - \phi_i\|_W \to 0$, $i = l, ..., n$. *Let* $\|f - f_k\|_W \to 0$ *and* $d(X, X_k) \to 0$. *Let*

(H1) *Any accumulation point of a sequence* $\{B^k\}$ *with* $B^k \in C_k$ *must be in* C, *and*

(H2) *For given* $B \in C$ *and* $\delta > 0$, *there is* $B^\delta \in C$ *with* $\|B - B^\delta\| \leqslant \delta$ *and a sequence* $\{B^k\} \to B^\delta$, $B^k \in C_k$.

Let A^k *be a best coefficient vector to* $f_k \in C(W)$ *on* X_k *by linear combinations of* $\{\phi_1^k, ..., \phi_n^k\}$ *with constraint* C_k. *Then* $\{A^k\}$ *has an accumulation point* A *and any accumulation point is best to* f *on* X *by linear combinations of* $\{\phi_1, ..., \phi_n\}$ *with constraint* C.

A special case of the above theorem with no constraints, that is, $C = C_k = n$-space, was obtained in [2].

Proof.

Remark. This proof is a straightforward elaboration of the proof of the corresponding result in [2].

Suppose $\{\|A^k\|\}$ is unbounded. Then we can assume without loss of generality that $\|A^k\| > k$. Define $B^k = A^k/\|A^k\|$, then $\|B^k\| = 1$ and $\{B^k\}$ has an accumulation point B, $\|B\| = 1$. Assume $\{B^k\} \to B$. From the linear independence of $\{\phi_1, ..., \phi_n\}$ on X, it follows that there exists $x \in X$ with $L(B)(x) \neq 0$. By continuity there exists K and $\delta > 0$ such that

$$|L^k(B^k)(y)| \geqslant |L(B)(x)|/2, \qquad k > K, \rho(x, y) \leqslant \delta.$$

There is a sequence $\{x_k\} \to x$, $x_k \in X_k$. There exists J such that for $k > J$, $\rho(x, x_k) < \delta$. For $k > \max\{J, K\}$,

$$|L^k(B^k)(x_k)| \geqslant |L(B)(x)|/2,$$

hence

$$|L^k(A^k)\,(x_k)| > k\,|L(B)\,(x)|/2.$$

As f_k is bounded on W, this implies that $\{|f_k(x_k) - L^k(A^k)(x_k)|\} \to \infty$. Now by (H2) there is $D \in C$ and a sequence of coefficient vectors $\{D^k\} \to D$ such that $D^k \in C_k$. Since A^k is best

$$|f_k(x_k) - L^k(A^k)(x_k)| \leqslant \|f_k - L^k(D^k)\|_W \leqslant \|f_k\|_W + \|L^k(D^k)\|_W.$$

We have a contradiction and $\{A^k\}$ is bounded, hence it has an accumulation point A. Assume without loss of generality that $\{A^k\} \to A$. By (H1), $A \in C$. Suppose there is $B \in C$, $\varepsilon > 0$ with

$$\|f - L(B)\|_X < \|f - L(A)\|_X - \varepsilon.$$

By taking δ sufficiently small, we get

$$\|f - L(B^\delta)\|_X < \|f - L(A)\|_X - \varepsilon$$

and $\{B^k\} \to B^\delta$, $B^k \in C_k$ by (H2).

We have

$$\|f_k - L^k(B_k)\|_k \to \|f - L(B^\delta)\|_X$$

$$\|f_k - L^k(A^k)\|_k \to \|f - L(A)\|_X;$$

hence for all k sufficiently large

$$\|f_k - L^k(B^k)\|_k < \|f_k - L^k(A^k)\|_k - \varepsilon/2,$$

contradicting optimality of A^k, and proving the theorem.

Remark. In the proof of the corresponding result in [2], the superscript k on the L's in the three above formulas was incorrectly omitted.

Remark. In case $C = C_1 = C_2 = C_n = \dots$, hypothesis (H1, H2) are automatically satisfied. To require that all coefficients lie in a fixed range, say all coefficients $a_i \geqslant 0$, is such a constraint.

We now apply our theory. First consider approximation with interpolation of function values. Let $\{x_1,\dots, x_m\}$ be a set of m distinct points of X. Let $\{x_1^k,\dots, x_m^k\} \in X_k$ and $\{(x_1^k,\dots, x_m^k)\} \to \{x_1,\dots, x_m\}$. The interpolatory constraint is to choose (for superscript s)

$$C_s = \{A : L^s(A)(x_i^s) = f_s(x_i^s),\ i = l,\dots, m\}.$$

C_s is closed for all s.

Assume that $n-m$ other points $\{x_{m+1},\ldots, x_n\}$ of X can be chosen such that $\{\phi_1,\ldots, \phi_n\}$ is a Chebyshev set on $\{x_1,\ldots, x_n\}$. This can always be done if it is a Chebyshev set on X. Choose $\{x^k_{m+1},\ldots, x^k_n\} \in X_k$, with $\{(x^k_{m+1},\ldots, x^k_n)\} \to (x_{m+1},\ldots, x_n)$. Now let $B \in C$ be given. It is a solution to the linear system with unknown A

$$L(A)(x_i) = L(B)(x_i), \qquad i = 1,\ldots, n. \tag{*}$$

As the matrix of the linear system is a generalized Vandermonde matrix, which is nonsingular, B is uniquely determined by (*). Next consider the linear system

$$\begin{aligned} L^{(k)}(A^k)(x^k_i) &= f_k(x^k_i), && i = 1,\ldots, m, \\ &= L(B)(x^k_i), && i = m+1,\ldots, n. \end{aligned}$$

By continuity (in the neighbourhood of a nonsingular case) of the solution of a linear system with respect to its matrix entries and right-hand side, $\{A^k\} \to B$ as $k \to \infty$. Hypothesis (H2) is verified. To verify hypothesis (H1),

$$L^k(A^k)(x^k_i) = f_k(x^k_i), \qquad i = 1,\ldots, m,$$

then $\{A^k\} \to A$ implies

$$L(A)(x_i) = f(x_i), \qquad i = 1,\ldots, m.$$

Next consider restricted range approximation with unequal restraints. Let μ, $\nu \in C(W)$, $\mu < \nu$. Let μ_k, $\nu_k \in C(W)$, $\mu_k < \nu_k$, and $\{\mu_k\} \to \mu$, $\{\nu_k\} \to \nu$. We have

$$C_s = \{A : \mu_s(x) \leqslant L^s(A)(x) \leqslant \nu_s(x), x \in X_s\}.$$

C_s is closed for all s. We make the additional assumption

ASSUMPTION. *C is nonempty and given $B \in C$ and $\delta > 0$, there is B^δ such that $\|B - B^\delta\| \leqslant \delta$ and*

$$\mu(x) < L(B^\delta)(x) < \nu(x), \qquad x \in X.$$

We establish hypothesis (H2) and (H1). Let $\{B^k\} \to B^\delta$, then for all k sufficiently large

$$\mu_k(x) \leqslant L^k(B^k)(x) \leqslant \nu_k(x), \qquad x \in X_k.$$

Thus hypothesis (H2) is satisfied. Next let $\{A^k\} \to A$, $A^k \in C_k$. We claim

$$\mu(x) \leqslant L(A)(x) \leqslant \nu(x), \qquad x \in X.$$

Suppose this is false; without loss of generality suppose

$$\mu(x) > L(A)(x) + \varepsilon.$$

Then for all k sufficiently large and $\{x_k\} \to x$ we have

$$\mu_k(x_k) > L^k(A^k)(x_k) + \varepsilon/2,$$

giving a contradiction. Thus hypothesis (H1) is verified.

A case of special interest is one-sided approximation from above or below, in which case we set one of the restraints μ, ν equal to f and drop the other (or set it to $\pm\infty$). If there is an approximant >0, our additional assumption is always satisfied.

The additional assumption we made may be necessary if we perturb bases, domains of approximation, or restraints.

EXAMPLE. Let $X = X_k = [0, 1]$. Let $\mu = 0$ and $\nu = +\infty$. Let $\phi_1(x) = x$ and $\phi_1^k(x) = x - (1/k)$. The only multiple of ϕ_1^k satisfying the constraint is the zero multiple.

EXAMPLE. Let $X = [0, 1]$ and $X_k = [-1/k, 1]$. Let $\mu = 0$ and $\nu = +\infty$. Let $\phi_1(x) = \phi_1^k(x) = x$. The only approximation satisfying the constraint on X_k is the zero approximation.

EXAMPLE. Let $X = X_k = [0, 1]$ and $\phi_1(x) = \phi_1^k(x) = x^2$. Let $\mu = 0$ and $\mu_k = x/k$. Let $\nu = +\infty$. 0 is in C, but C_k is empty.

An extension of restricted range approximation is approximation with one or several derivatives of the approximation having restricted ranges, say

$$\mu^j \leqslant L^{(j)}(A) \leqslant \nu^j, \qquad j \in J.$$

The additional assumption in this case is that C is nonempty and given $B \in C$ and $\delta > 0$, there is B^δ with $\|B - B^\delta\| < \delta$ and

$$\mu^j < L^{(j)}(B^\delta) < \nu^j, \qquad j \in J.$$

The perturbation result is proven as for the ordinary restricted range problem. Monotone approximation (treated next) is often converted to $L'(A) \geqslant 0$ ($\leqslant 0$) and convex approximation (treated shortly) is often converted to $L''(A) \geqslant 0$.

In the case we require $L^{(j)}(A) \geqslant 0$ for a single j, the additional hypothesis is satisfied if there exists D such that $L^{(j)}(D) > 0$ on an open set containing X.

Another possible constraint in real approximation on subsets of the real line is that approximations be monotone increasing (decreasing). A general perturbation result is not possible for this constraint if bases or domains of approximation are allowed to vary.

EXAMPLE. Let $X = X_k = [-1, 1]$ and $f(x) = f_k(x) = x$. Let $\phi_1 = \phi_1^k = 1$. Let $\phi_2(x) = x$ and $\{\phi_2^k\}$ be a sequence of nonmonotone functions converging uniformly to ϕ_2. The approximation ϕ_2 is equal to f and therefore uniquely best among linear combinations of $\{\phi_1, \phi_2\}$. As only the constants are monotone among linear combinations of $\{\phi_1^k, \phi_2^k\}$, the best monotone approximation to f by these must be the best constant approximation, namely zero.

EXAMPLE. Let $X = [0, 1]$ and $X_k = [-1/k, 1]$. Let $f(x) = f_k(x) = 2x - 1$. Let $\phi_1 = 1$ and $\phi_2(x) = x^2$. All approximations are monotone on X, but only constants are monotone on X_k. By the classical theory of approximation by a Haar subspace on an interval, $L(A^*)$ best on X implies $L(A^*)$ is unique and $f - L(A^*, \cdot)$ alternates twice on $[0, 1]$ with amplitude >0. Let $L^k(A^k)$ be a best constant approximation to f on X_k, then $f - L^k(A^k)$ alternates once and is monotone.

If we restrict our attention to fixed bases and approximations on subsets (i.e., $X_k \subset X$), a perturbation result holds. Assume the constraint is that approximants be monotone increasing on the domain of approximation.

Let $\{A^k\}$ be a sequence of coefficient vectors such that $L(A^k, \cdot)$ is monotone on X_k and $\{A^k\} \to A$. Suppose $L(A, \cdot)$ is not monotone on X, then there is $x < y$ with $L(A)(x) > L(A)(y) - \varepsilon$. Let $\{x_k\} \to x$, $x_k \in X_k$ and $\{y_k\} \to y$, $y_k \in X_k$. Then for all k sufficiently large, $L(A^k)(x_k) > L(A^k)(y_k) - \varepsilon/2$ and we have a contradiction. Hence hypothesis (H1) holds. Next let $L(B)$ be monotone on X, then $L(B)$ is monotone on any subset and hypothesis (H2) holds. We can, therefore, apply our generalized perturbation result. A generalization of the constraint is being comonotone [1] and the above result generalizes.

If $\{\phi_1, ..., \phi_n\}$ are monotone increasing and $a_i \geqslant 0$ for $i = 1, ..., n$, the linear combination $L(A)$ is monotone increasing. Thus if bases are monotone, we might replace the monotonicity constraint by the stronger constraint $a_i \geqslant 0$, which leads to a simple perturbation theory by a remark preceding examples of constraints.

A result related to the main result of this paper is given in Appendix A of the dissertation of Levasseur [6].

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Printed in Belgium